ON THE THEORY OF OPTIMUM CONTROL

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In the present paper problems of optimum (in the sense of high-speed action) control of systems containing a linear basic part and corresponding to certain basic types of restrictions on the controlling actions are considered. Limit passages in the solutions are discussed which correspond to the passages from one type of restrictions to another. On the basis of these limit passages, approximate methods are described for the computation of optimum trajectories and the construction of optimum systems.

1. In this section the basic problems of optimum control considered in the present paper are formulated.

Let the behavior of the phase coordinates $x_i(t)$ $(i = 1, \ldots, n)$ of a control system be described by the differential equations

$$\frac{dx}{dt} = Ax + b\eta + e^1\xi^1 + \ldots + e^{n-1}\xi^{n-1} \tag{1.1}$$

where

$$x = \{x_1, \ldots, x_n\}, \quad b = \{b_1, \ldots, b_n\}, \quad e^\alpha = \{e_1^\alpha, \ldots, e_n^\alpha\} \quad (\alpha = 1, \ldots, n-1)$$

are n-dimensional vectors, A is an $n \times n$ matrix with constant elements a_{ij}, and η, ξ_α $(\alpha = 1, \ldots, n-1)$ are scalar functions.

Given the initial conditions $x_0 = \{x_{10}, \ldots, x_{n0}\}$, it is required to find the functions η_0, ξ_0^α (optimum control) in such a way that the point $x(t) = x(x_0, t, \eta_0, \{\xi_0^\alpha\})$, moving along a trajectory of system (1.1), where $\eta = \eta_0$, $\xi^\alpha = \xi_0^\alpha$, reaches the origin of coordinates $x = 0$ in the shortest possible time $t = T^0$ (T^0 being the optimum control time). It is assumed that the (admissible) functions $\eta(t)$, $\xi^\alpha(t)$ $(\alpha = 1, \ldots, n-1)$ and the coefficients of system (1.1) are restricted by one of the following conditions:

Problem I. The coefficients $e_\beta^\alpha = 0$ $(\alpha = 1, \ldots, n-1;\ \beta = 1, \ldots, n)$

and $\eta(t)$ is a piecewise smooth function restricted by the condition

$$|\eta(t)| \leqslant 1, \qquad 0 \leqslant t \leqslant T^\circ \tag{1.2}$$

Problem II. The functions $\eta(t)$, $\xi^a(t)$ $(a = 1, \ldots, n-1)$ are continuous and satisfy the condition

$$\left(\eta^2(t) + \sum_{\alpha=1}^{n-1} [\xi^\alpha(t)]^2\right) \leqslant 1, \qquad 0 \leqslant t \leqslant T^\circ \tag{1.3}$$

Problem III. The function $\eta(t) = d\zeta(t)$ is the Stieltjes differential of another function $\zeta(t)$ which is bounded and restricted by the condition

$$\int_0^{T^\circ} |d\zeta(t)| \leqslant 1 \tag{1.4}$$

the coefficients $e_\beta^a = 0$ $(a = 1, \ldots, n-1;\ \beta = 1, \ldots, n)$.

Problem IV. The coefficients $e_\beta^a = 0$ $(a = 1, \ldots, n-1;\ \beta = 1, \ldots, n)$ and $\eta(t)$ is a continuous function restricted by the condition

$$\left(\int_0^{T^\circ} |\eta(t)|^p dt\right)^{1/p} \leqslant 1, \qquad 1 < p < \infty \tag{1.5}$$

Problems I, II and IV are problems of optimum control with one steering function η. The condition (1.2) corresponds to a restriction on a controlling action (force, current, stress, and so on) at each instant t in the transitional process $0 \leqslant t \leqslant T^0$. The condition (1.4) is a restriction on the impulses of the controlling quantities*. To the condition (1.5) for $p = 2$ corresponds a restriction on the energy (mean power) of the control actions. It is interesting to consider this condition also for other values of $p \in [1, \infty)$ from the point of view of limit passages to the Problems I $(p \to \infty)$ and III $(p \to 1)$.

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* Also another problem which occurs in control theory can be reduced to the problem of type III, namely the problem of construction of optimum control actions for a system, described by the equation

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_n x = \sum_{\alpha,\beta=1}^{n} e_{\alpha\beta} \xi^\alpha \xi^\beta \tag{1.6}$$

where the admissible control functions ξ^a $(a = 1, \ldots, n)$ are restricted by the condition

$$\int_0^{T^\circ} \left(\sum_{\alpha,\beta=1}^{n} c_{\alpha\beta} \xi^\alpha(t) \xi^\beta(t)\right) dt \leqslant 1 \tag{1.7}$$

and $\sum_{\alpha,\beta=1}^{n} c^{\alpha\beta} \xi^\alpha \xi^\beta$ is a positive definite quadratic form.

Problem II is a problem of optimum control with n controlling actions restricted by the condition (1.3) at each instant t in the transitional process $0 \leqslant t \leqslant T^0$. For sufficiently general assumptions Problem II assumes smooth solutions $\eta_0(t)$, $\xi_0^a(t)$ contrary to Problem I, for which, as a rule, the optimum steering $\eta_0(t)$ is a discontinuous function. It is also of interest to investigate the limit passage from Problem II to Problem I for $e_\beta^a \to 0$ $(a = 1, \ldots, n-1;\ \beta = 1, \ldots, n)$. After the justification of such a limit passage, by means of which we can approximate Problem I by Problem II with small e_β^a, it is possible to construct an approximate method for the solution of Problem I in terms of continuous optimum controlling functions $\eta_0(t)$ of Problem II.

The method applied below for the investigation of Problems I, II and IV can also be used in the case of several controlling actions $\eta^a(t)$ $(a = 1, \ldots, r)$. However, in the present paper for the solution of Problems I, II and IV we shall restrict ourselves to a single controlling action $\eta(t)$.

Problems of optimum control in the sense of rapid action have been considered by many authors (see, for example, [1-4]). In this paper we shall restrict ourselves to the problem for which the basic part of system (1.1) [for $\eta = 0$, $\xi^a = 0$ $(a = 1, \ldots, n-1)$] is linear. Problems I to IV, and similar ones which can be reduced to them, can be considered from a unified point of view, indicated in the paper [5], provided these problems are reduced to a single problem of functional analysis (L-problem in abstract space, article IV, [6]), considered for each of the Problems I to IV in a specially selected functional space.*

Investigating the Problems I to IV, we shall restrict ourselves to the case where the roots $\lambda_i (i = 1, \ldots, n)$ of the characteristic equation

$$|A - \lambda E|_1^n = 0 \tag{1.8}$$

on the basic linear system

$$dx/dt = Ax \tag{1.9}$$

* The classes of functions defined in the formulations of Problems I to IV do not coincide with the classes of functions corresponding to the functional spaces selected below. However, for further exposition this is not essential, since the minima sought below are reached on functions $\eta(t)$ and $\xi^a(t)$ from the classes defined in the formulation of Problems I to IV.

satisfy the condition*

$$\operatorname{Re}\lambda_i < 0 \qquad (i = 1, \ldots, n) \tag{1.10}$$

and the vectors

$$b,\ Ab, \ldots, A^{n-1}b \tag{1.11}$$

are linearly independent, i.e.

$$l_1 b + l_2 Ab + \ldots + l_n A^{n-1} b \neq 0 \quad \text{for } l_1^2 + \ldots + l_n^2 \neq 0 \tag{1.12}$$

holds.

Let us remark in conclusion of the formulation of the problem that for the synthesis of regulating systems it is important to find the optimum controlling quantities η and ξ^{α} $(\alpha = 1, \ldots, n-1)$ not only (and not so much) as functions of time t, but also as functions of the phase coordinates x_i of the system. We shall denote these functions of the coordinates symbolically by η and ξ^{α} [in a detailed writing in the form $\eta(x_1, \ldots, x_n)$, $\xi^{\alpha}(x_1, \ldots, x_n)$].

2. In this section the reduction of the Problems I to IV to the L-problem (article IV of [6]) in standard functional spaces is described.**

Let $F(t)$ denote the matrix of the fundamental system of solutions for the equations (1.9). We shall denote by the symbols $\{F(t)\}_{ij}$ and the elements of the reciprocal matrix $F^{-1}(t)$ - by the symbols $f_{ij}(t)$. The solutions $x(x_0, t, \eta, \{\xi^{\alpha}\})$ of the nonhomogeneous system (1.1) must be calculated by the Cauchy formula [7]

$$x(x_0, t, \eta, \{\xi^{\alpha}\}) = F(t)x_0 + \int_0^t F(t)F^{-1}(\tau)\Big[b\eta(\tau) + \sum_{\alpha=1}^{n} e^{\alpha}\xi^{\alpha}(\tau)\Big]\,d\tau \tag{2.1}$$

For

$$t = T^{\circ}, \qquad \eta = \eta_0, \qquad \xi^{\alpha} = \xi_0^{\alpha} \tag{2.2}$$

according to the conditions of the Problems I to IV the equality $x(x_0, T^0, \eta_0, \{\xi_0^{\alpha}\}) = 0$ must be satisfied, i.e. after the substitution of (2.2) into (2.1) and multiplication of this equality by $F^{-1}(T^0)$ we obtain the equality

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* Condition (1.10) assures the possibility of optimum steering into the point $x = 0$ for all arbitrary large initial values x_{i0}. If this condition is not satisfied, the arguments mentioned in the article remain in force for a certain (in general, finite) region of the space $\{x_{i0}\}$.

** Section 2 has the goal to describe the problems, part of which was considered earlier from a unified point of view.

$$-x_0 = \int_0^{T^0} F^{-1}(\tau)\left[b\eta(\tau) + \sum_{\alpha=1}^{n-1} e^{\alpha}\xi^{\alpha}(\tau)\right] d\tau \tag{2.3}$$

Thus the optimum time of control for each of the Problems I to IV will be the smallest of the numbers T^0, satisfying the conditions

$$-x_{i0} = \int_0^{T} \left\{h_i(\tau)\eta(\tau) + \sum_{\alpha=1}^{n-1} g_i^{\alpha}(\tau)\xi^{\alpha}(\tau)\right\} d\tau \quad (i = 1, \ldots, n) \tag{2.4}$$

The functions $h_i(\tau)$ and $g_i^{\alpha}(\tau)$ are given by the formulas

$$h_i(\tau) = \sum_{k=1}^{n} f_{ik}(\tau) b_k \qquad (i = 1, \ldots, n) \tag{2.5}$$

$$g_i^{\alpha}(\tau) = \sum_{k=1}^{n} f_{ik}(\tau) e_k^{\alpha} \qquad (i = 1, \ldots, n;\ \alpha = 1, \ldots, n-1) \tag{2.6}$$

and the functions $\eta(t)$ and $\xi^{\alpha}(t)$ are restricted by one of the conditions (1.2) to (1.5), corresponding to the Problems I to IV.

Consider the functions $h_i(t)$ and $g_i^{\alpha}(t)$ $(0 \leqslant t \leqslant T)$ as the elements of the following functional spaces* (L, C, L_q) [14] [(B I) - (B IV) corresponding to the Problems I to IV] :

(1) space (B I), the elements h of which are the functions $h(t)$ $(0 \leqslant t \leqslant T)$ with the norm

$$\|h\| = \int_0^{T} |h(\tau)|\, d\tau \tag{2.7}$$

(2) space (B II), the elements $\{h, g\}$ of which are the vector functions $h(t)$, $g^{\alpha}(t)$ $(\alpha = 1, \ldots, n \times 1)$, $(0 \leqslant t \leqslant T)$ with the norm

$$\|\{h, g\}\| = \int_0^{T} \left\{h^2(\tau) + \sum_{\alpha=1}^{n-1} [g^{\alpha}(\tau)]^2\right\}^{1/2} d\tau \tag{2.8}$$

(3) space (B III), the elements h of which are the functions $h(t)$ $(0 \leqslant t \leqslant T)$ with the norm

$$\|h\| = \sup |h(t)| \qquad \text{for } 0 \leqslant t \leqslant T \tag{2.9}$$

(4) space (B IV), the elements h of which are the functions $h(t)$ $(0 \leqslant t \leqslant T)$ with the norm

$$\|h\| = \left(\int_0^{T} |h(\tau)|^q\, d\tau\right)^{1/q} \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right) \tag{2.10}$$

* See the footnote on p. 901.

Also consider the functions $\eta(t)$ and $\xi^a(t)$ as the elements of the following conjugate spaces (M, C*, L_p):

(1*) space (B* I), the elements η of which are the functions $\eta(t)$ with the norm

$$\|\eta\| = \sup|\eta(t)| \quad \text{for } t \in [0,T]\setminus E \tag{2.11}$$

(2*) space (B* II), the elements $\{\eta, \xi\}$ of which are the vector functions $\eta(t)$, $\xi^a(t)$ $(a = 1, \ldots, n-1)$ with the norm

$$\|\{\eta, \xi\}\| = \sup\left[\eta^2(t) + \sum_{\alpha=1}^{n-1}[\xi^\alpha(t)]^2\right]^{1/2} \quad \text{for } 0 \leqslant t \leqslant T \tag{2.12}$$

(3*) space (B* III), the elements η of which are the functions $\eta = d\zeta$ with the norm

$$\|\eta(t)\| = \int_0^T |\eta(t)|\,dt \tag{2.13}$$

(4*) space (B* IV), the elements η of which are the functions $\eta(t)$ with the norm

$$\|\eta(t)\| = \left(\int_0^T |\eta(t)|^p\,dt\right) \tag{2.14}$$

Then the functions $\eta(t)$, $\xi^a(t)$ determine linear functionals ϕ on the elements of (B I) to (B IV), i.e.

$$\varphi[h] = \int_0^T h(\tau)\eta(\tau)\,d\tau \quad (\text{I, IV}), \qquad \varphi[h] = \int_0^T h(\tau)\,d\zeta(\tau) \quad (\text{III}) \tag{2.15}$$

holds in the spaces (B I), (B III), (B IV) and

$$\varphi[\{h, g\}] = \int_0^T \left(h(\tau)\eta(\tau) + \sum_{\alpha=1}^{n-1} g^\alpha(\tau)\xi^\alpha(\tau)\right) d\tau \tag{2.16}$$

in the space (B II). The norms of the functionals ϕ are determined by the formulas (2.11) to (2.14), respectively. In this way each of the Problems I to IV is reduced to the following problem: to find the smallest number T and a linear functional ϕ in the corresponding functional space in such a way that

$$\varphi[h_\beta] = -x_{\beta 0} \qquad \left(h_\beta = \sum_{\gamma=1}^{n} f_{\beta\gamma}(\tau)\, b_\gamma,\ \beta = 1, \ldots, n\right) \tag{2.17}$$

$$\|\varphi\| \leqslant 1 \tag{2.18}$$

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* See the footnote on p. 901.

hold for Problems I, III, IV and

$$\varphi[\{h_\beta, g_\beta\}] = -x_{\beta 0} \qquad (\beta = 1, \ldots, n) \tag{2.19}$$

$$h_\beta = \sum_{\gamma=1}^{n} f_{\beta\gamma}(\tau)\, b_\gamma, \qquad g_\beta^\alpha = \sum_{\gamma=1}^{n} f_{\beta\gamma}(\tau)\, e_\gamma^\alpha \tag{2.20}$$

for Problem II.

For a given T the problems (2.17), (2.18) [or (2.18) to (2.20)] have a solution if, and only if, (see [6])

$$\min \|(l \cdot h)\| = \lambda(T) \geqslant 1, \qquad (x_0 \cdot l) = -1 \tag{2.21}$$

or, respectively,

$$\min \|(l \cdot \{h, g\})\| = \lambda(T) \geqslant 1, \qquad (x_0 \cdot l) = -1 \tag{2.22}$$

are satisfied.

In order to abbreviate writing the following notations are used

$$(l \cdot h) = \sum_{\beta=1}^{n} l_\beta h_\beta(\tau), \quad (l \cdot x_0) = \sum_{\beta=1}^{n} l_\beta x_{\beta 0} \tag{2.23}$$

$(l \cdot \{h, g\})$, being a vector function, is an element of the space (B II), and has the components

$$\sum_{\beta=1}^{n} l_\beta h_\beta(\tau), \quad \sum_{\beta=1}^{n} l_\beta g_\beta^1(\tau), \ldots, \quad \sum_{\beta=1}^{n} l_\beta g_\beta^{n-1}(\tau) \tag{2.24}$$

By virtue of our restrictions the quantity $\lambda(T)$ is a monotonically increasing function of the argument T, satisfying the condition*

$$\lim_{T\to\infty} \lambda(T) = \infty \tag{2.25}$$

$$\lim_{T\to 0} \lambda(T) = 0 \text{ (I, II, IV)} \tag{2.26}$$

Consequently, for each $x = x_0$ the problem has a solution for which the optimum control time T^0 is to be computed from the equation

$$\min \|(l \cdot h)\| = \lambda(T) = 1, \qquad (x_0 \cdot l) = -1 \tag{2.27}$$

or from

$$\min \|(l \cdot \{h, g\})\| = \lambda(T) = 1, \qquad (x_0 \cdot l) = -1 \tag{2.28}$$

respectively.

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* Under the conditions (1.12) the equality

$$\|(l \cdot h)\| = 0 \text{ for } \Sigma l_\beta^2 \neq 0$$

is possible only for particular isolated values of $t \in [0, T]$ (see [8,9]) and (2.25) is an obvious consequence of (1.10).

According to the results of the book [6] the functional ϕ (or, what is the same, the optimum controlling functions $\eta(t)$ or $\eta(t), \xi^{\alpha}(t)$)) is to be determined from the condition that the element

$$h = \sum_{\beta=1}^{n} l_\beta{}^\circ h_\beta(t)$$

or

$$(l^\circ \cdot \{h, g\}) = \left\{\sum_{\beta=1}^{n} l_\beta{}^\circ h_\beta(t),\ \sum_{\beta=1}^{n} l_\beta{}^\circ g_\beta{}^1(t), \ldots, \sum_{\beta=1}^{n} l_\beta{}^\circ g_\beta{}^{n-1}(t)\right\}$$

$l^0 = \{l_\beta{}^0\}$ being the solution of the problems (2.21), (2.22), is an extremum for the respective functional, i.e. the equality

$$\|\varphi\|\|(l^\circ \cdot h)\| = |\varphi[(l^\circ \cdot h)]|$$

or, respectively, the equality

$$\|\varphi\|\|(l^\circ \{h, g\})\| = |\varphi[(l^\circ \cdot \{h, g\})]|$$

is satisfied.

From these general results concerning the problems I to IV we obtain the following deductions:

1. For Problem I the optimum control has the form

$$\eta_0(t) = \operatorname{sign}\left(\sum_{\beta=1}^{n} l_\beta{}^\circ h_\beta(t)\right) \tag{2.29}$$

where the numbers $l_\beta{}^0$ $(\beta = 1, \ldots, n)$ are the solutions of

$$\min \int_0^{T^\circ} \left|\sum_{\beta=1}^{n} l_\beta h_\beta(\tau)\right| d\tau = 1, \qquad \sum_{\beta=1}^{n} l_\beta x_\beta = -1 \tag{2.30}$$

2. For Problem II the optimum control has the form

$$\eta_0(t) = \frac{\sum_{\beta=1}^{n} l_\beta{}^\circ h_\beta(t)}{\left[\left(\sum_{\beta=1}^{n} l_\beta{}^\circ h_\beta(t)\right)^2 + \left(\sum_{\beta=1}^{n} l_\beta{}^\circ g_\beta{}^1(t)\right)^2 + \ldots + \left(\sum_{\beta=1}^{n} l_\beta{}^\circ g_\beta{}^{n-1}(t)\right)^2\right]^{1/2}}$$

$$\xi_0{}^\alpha(t) = \frac{\sum_{\beta=1}^{n} l_\beta{}^\circ g_\beta^\alpha(t)}{\left[\left(\sum_{\beta=1}^{n} l_\beta{}^\circ h_\beta(t)\right)^2 + \left(\sum_{\beta=1}^{n} l_\beta{}^\circ g_\beta{}^1(t)\right)^2 + \ldots + \left(\sum_{\beta=1}^{n} l_\beta{}^\circ g_\beta{}^{n-1}(t)\right)^2\right]^{1/2}} \tag{2.31}$$

$$(\alpha = 1, \ldots, n-1)$$

where the numbers $l_\beta{}^0$ $(\beta = 1, \ldots, n)$ are the solutions of

$$\min \int_0^{T^\circ} \left[\left(\sum_{\beta=1}^n l_\beta h_\beta(\tau)\right)^2 + \left(\sum_{\beta=1}^n l_\beta g_\beta{}^1(\tau)\right)^2 + \ldots + \left(\sum_{\beta=1}^n l_\beta g_\beta{}^{n-1}(t)\right)^2\right]^{1/2} d\tau = 1$$

$$\sum_{\beta=1}^n l_\beta x_{\beta 0} = -1 \qquad (2.32)$$

3. For Problem III the optimum control has the form

$$\eta_0(t) = \sum_{\gamma=1}^r \mu_\gamma \delta(t - t_\gamma), \qquad \sum_{\gamma=1}^n |\mu_\gamma| = 1 \qquad (2.33)$$

where $\delta(t)$ denotes the impulsive δ-function, t_γ the instants of time at which the function $|\Sigma l_\beta{}^0 h_\beta(t)|$ assumes its largest value on the segment $[0, T^0]$, and the numbers $l_\beta{}^0$ are the solutions of

$$\min\left(\max\left|\sum_{\beta=1}^n l_\beta h_\beta(t)\right| \quad \text{for} \quad 0 \leqslant t \leqslant T^\circ\right) = 1 \qquad \left(\sum_{\beta=1}^n l_\beta x_{\beta 0} = -1\right) \qquad (2.34)$$

4. For Problem IV the optimum control has the form

$$\eta_0(t) = \operatorname{sign}\left(\sum_{\beta=1}^n l_\beta h_\beta(t)\right)\left|\sum_{\beta=1}^n l_\beta h_\beta(\tau)\right|^{q-1} \qquad (2.35)$$

where the numbers $l_\beta{}^0$ are the solutions of

$$\min \int_0^{T^\circ} \left|\sum_{\beta=1}^n l_\beta h_\beta(\tau)\right|^q d\tau = 1 \qquad \left(\sum_{\beta=1}^n x_{\beta 0} l_\beta = -1\right) \qquad (2.36)$$

3. Consideration of Problems I to IV from the general point of view, as described in Section 2, allows us to investigate the limit passages in the solutions of these problems when passing from one type of problem to another type. Since Problem I is the most common one, it is interesting to investigate limit passages from other "smooth" problems to this problem, which has discontinuous solutions. In this paper, we shall investigate in all detail Problem II and the limit passage from Problem II to Problem I. This passage is interesting, in particular, because the solution of Problem II can be reduced to the solution of a certain ordinary differential equation, and also because of the fact that Problem II admits a smooth Liapunov function as it will be shown below.

We shall assume that the coefficient matrix

$$\begin{Vmatrix} b_1 & e_1{}^1 & \ldots & e_1{}^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ b_n & e_n{}^1 & \ldots & e_n{}^{n-1} \end{Vmatrix} \qquad (3.1)$$

is nonsingular.

In this section we shall establish that the optimum control time T^0 for Problem II is a continuously differentiable function of the coordinates x_{i0} of the initial point x_0.

Consider a system of differential equations of a more general type than (1.1), namely

$$dx/dt = \vartheta Ax + b\eta + e^1\xi^1 + \ldots + e^{n-1}\xi^{n-1} \tag{3.2}$$

where θ is a certain parameter which assumes non-negative values. We shall denote the optimum control time T^0 and the optimum control functions η_0, ξ_0^{α} for Problems I and II by virtue of system (3.2) by the symbols

$$T_{\rm I}^{\circ}(x_1, \ldots, x_n, \vartheta), \quad T_{\rm II}^{\circ}(x_1, \ldots, x_n, \vartheta), \quad \eta_{0\rm I}(x_1, \ldots, x_n, \vartheta)$$
$$\eta_{0\rm II}(x_1, \ldots, x_n, \vartheta), \qquad \xi_0^{\alpha}(x_1, \ldots, x_n, \vartheta)$$

(or, briefly by $T_{\rm I}^0(x, \theta)$, $T_{\rm II}^0(x, \theta)$ and so on). The indices I and II will be dropped if ambiguity is not likely to occur.

Theorem 3.1. Assume that the matrix of the coefficients (3.1) is nonsingular, i.e. the determinant

$$\begin{vmatrix} b_1 & e_1^1 & \ldots & e_1^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ b_n & e_n^1 & \ldots & e_n^{n-1} \end{vmatrix} \neq 0 \tag{3.3}$$

and that the condition (1.10) is satisfied. Then for Problem II the optimum control time T^0, being a function $T^0(x_1, \ldots, x_n, \theta)$ of the coordinates x_β of the initial point and a parameter θ, has partial derivatives of any order with respect to all arguments for all $x \neq 0$, $\theta > 0$.

Proof. According to the results of Section 2 in the case of Problem II the optimum control time $T^0(x_1, \ldots, x_n, \theta)$ for the system (3.2) is to be determined from the equation

$$\min\int_0^T \Big[\Big(\sum_{\beta=1}^n l_\beta \sum_{\gamma=1}^n f_{\beta\gamma}(t, \vartheta) b_\gamma\Big)^2 + \sum_{\alpha=1}^{n-1}\Big(\sum_{\beta=1}^n l_\beta \sum_{\gamma=1}^n f_{\beta\gamma}(t, \vartheta) e_\gamma^{\alpha}\Big)^2\Big]^{1/2} dt = 1$$

$$\text{for} \sum_{\beta=1}^n l_\beta x_\beta = -1 \tag{3.4}$$

where the symbols $f_{\beta\gamma}(t, \theta)$ denote the elements of the matrix $F^{-1}(t, \theta)$, this matrix being the reciprocal of the fundamental matrix $F(t, \theta)$ of the solutions of the system

$$dx / dt = \vartheta A x \tag{3.5}$$

Denote the left-hand side of equation (3.4) by the symbol $\lambda(x_1, \ldots, x_n, T, \theta)$. First of all let us note that for fixed values of $x_1, \ldots, x_n, \theta$ by virtue of the nonsingularity of the matrices $F^{-1}(t, \theta)$ and (3.1) the quantity λ is a monotonic increasing function of T. The existence and uniqueness of the solution of equation (3.4) were established above in Section 2, starting from general results with respect to the L-problem. Therefore, only the differentiability of the function T^0 remains to be proved. For this purpose, by virtue of well-known theorems on implicit functions, it is sufficient to verify that the function $\lambda(x_1, \ldots, x_n, T, \theta)$ possesses continuous partial derivatives of all orders with respect to $x_1, \ldots, x_n, T, \theta$ and that the condition

$$\partial\lambda / \partial T \neq 0 \tag{3.6}$$

holds.

Let us show first that the quantities l_β^0, which assign a minimum to the integral in (3.4), are continuous functions of $x_1, \ldots, x_n, T, \theta$ which can be differentiated an arbitrary number of times.

The fact that for every fixed T the minimum of the integral in (3.4) is actually reached for certain values of $l_\beta = l_\beta^0$, i.e. the existence of numbers l_β^0 which solve the problem is proved in the general case of the L-problem in the book [6].

Since for $\Sigma\, l_\beta^2 \neq 0$ the expression under the square-root sign in (3.4) cannot vanish, then for $\Sigma\, x^2 \neq 0$ the minimum of the integral in (3.4) can be found according to well-known rules of variational calculus.

In order to be more specific, assume that $x_1 \neq 0$. Then, using the condition $l_1 x_1 + \ldots + l_n x_n = -1$ to express l_1 in terms of the remaining $l_i (i = 2, \ldots, n)$ and substituting this expression in (3.4), we obtain

$$\lambda(x_1, \ldots, x_n, T, \vartheta) = \min_{l_2, \ldots, l_n} \gamma(x_1, \ldots, x_n, T, \vartheta, l_2, \ldots, l_n) \tag{3.7}$$

Here γ is a well-determined expression in terms of $x_1, \ldots, x_n, T, \theta, l_2, \ldots, l_n$, obtained from (3.4) by means of substitution

$$l_1 = \frac{1}{x_1}(-1 - l_2 x_2 - \ldots - l_n x_n) \tag{3.8}$$

Because of the cumbersome nature of this expression we shall not write it out here.

The numbers l_β^0 are determined by the equations

$$\partial\gamma / \partial l_\beta = 0 \qquad (\beta = 2, \ldots, n) \tag{3.9}$$

and equation (3.8). These numbers will be continuously differentiable functions of x_β, θ, T provided the corresponding functional determinant is different from zero, i.e.

$$\left| \frac{\partial^2\gamma}{\partial l_\beta \, \partial l_\alpha} \right|_2^n \neq 0 \tag{3.10}$$

holds. This is so, since the integral which determines the quantity γ, can be differentiated with respect to all the parameters x_β, T, θ, l_2, ..., l_n an arbitrary number of times (the existence of the derivatives of the elements $f_{\alpha\beta}(t, \theta)$ with respect to the parameter θ follows from well-known theorems on differentiation of the solutions of system (3.5) according to the parameter θ [11]).

For the proof of the inequality (3.10) it is sufficient to remark that the quadratic form

$$w(z_2, \ldots, z_n) = \sum_{\alpha, \beta=2}^{n} \frac{\partial^2\gamma}{\partial l_\alpha \, \partial l_\beta} z_\alpha z_\beta \tag{3.11}$$

is positive definite. The verification of this last condition follows easily from geometrical considerations. The analytical proof, however, requires cumbersome writing and will be omitted here. Hence, we can consider as established the fact that the quantities l_β^0 are functions of the arguments x_β, T, θ, having continuous derivatives of all orders.

Next we conclude that the function $\lambda(x_1, \ldots, x_n, T, \theta)$ has continuous partial derivatives of an arbitrary order with respect to all the arguments, since as a consequence of the differentiability of the numbers l_β^0 the integral, determining λ, can be differentiated with respect to all the parameters an arbitrary number of times.

Let us prove that inequality (3.6) is satisfied. Denote by T_0 any fixed value of T and by $l_\beta^0(T_0)$ $(\beta = 2, \ldots, n)$ the solutions of the problem (3.7), corresponding to this value of T. It is obvious that for $\Delta T > 0$ we have

$$\lambda(T_0 - \Delta T) = \min_{l_2, \ldots, l_n} \gamma(x_1, \ldots, x_n, T_0 - \Delta T, \vartheta, l_2, \ldots, l_n) \leqslant$$
$$\leqslant \gamma(x_1, \ldots, x_n, T_0 - \Delta T, \vartheta, l_2^\circ(T_0), \ldots, l_n^\circ(T_0))$$

i.e.

$$\left(\frac{\partial\lambda}{\partial T}\right)_{T_0} \geqslant \lim_{\Delta T \to 0} \frac{\gamma(x_\beta, T_0, \vartheta, l_\alpha^\circ(T_0)) - \gamma(x_\beta, T_0 - \Delta T, \vartheta\, l_\alpha^\circ(T_0))}{\Delta T} > 0$$

This proves the validity of inequality (3.6), Hence, the theorem is proved.

Remark. The arguments remain in force also in the case when $e_\beta{}^a$ are functions of θ.

4. In this section it is shown that in the case of Problem II the optimum controlling quantities $\eta_0(x_1, \ldots, x_n, \theta)$, $\xi_0{}^a(x_1, \ldots, x_n, \theta)$ $(a = 1, \ldots, n-1)$ for the system (3.2) are continuously differentiable functions of their arguments. Before we pass to the proof of this assertion, let us introduce certain concepts concerning the application of the method of Liapunov functions to the problem under consideration.

Consider anew the Problems I and II for the system (1.1).

Assume that the function $T^0(x_1, \ldots, x_n)$, being the optimum control time, is known and is continuously differentiable in a neighborhood of the point $(x_1, \ldots, x_n)$. It is obvious that if we replace $x_1, \ldots, x_n$ by the solutions $x_\beta(x_0, t, \eta_0, \{\xi_0{}^a\})$ $(\beta = 1, \ldots, n)$, then the total derivative of the function T^0 with respect to time t along the optimum trajectory must satisfy the equality

$$dT^\circ / dt = -1 \tag{4.1}$$

or, written out in full,

$$\frac{dT^\circ}{dt} = \sum_{\mu,\,\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta}\Big(a_{\beta\mu} x_\mu + b_\beta \eta_0(x) + \sum_{\alpha=1}^{n-1} e_\beta{}^\alpha \xi_0{}^\alpha(x)\Big) = -1$$

Moreover, the optimum control functions $\eta_0(x)$, $\xi_0{}^a(x)$ have the property that on the set of admissible functions the quantity

$$\sum_{\mu,\,\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta}\Big(a_{\beta\mu} x_\mu + b_\beta \eta(x) + \sum_{\alpha=1}^{n-1} e_\beta{}^\alpha \xi^\alpha(x)\Big)$$

assumes a minimum* only for these optimum control functions η_0, $\xi_0{}^a$. In this way the quantity $T^0(x)$ plays here the role of a Liapunov function. Let us explain this fact in all details. Assume that

$$\frac{dx}{dt} = Ax + b\eta_0(x) + \sum_{\alpha=1}^{n-1} e^\alpha \xi_0{}^\alpha(x) \tag{4.2}$$

is the optimum system obtained from system (1.1) for $\eta = \eta_0(x)$, $\xi^a = \xi_0{}^a(x)$. The origin of the coordinates $x = 0$ will be an asymptotically stable solution of system (4.2) with respect to arbitrary initial per-

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* The application of this reasoning to the case considered here corresponds to the general method of investigating problems of optimum control, elaborated by Iu.M. Repin on the basis of methods of dynamic programming.

turbations x_0 (with the singularity that $x(x_0, t, \eta_0, \{\xi_0^a\}) \to 0$ for $t \to T^0(x_0)$ and not for $t \to \infty$ as is usual in problems of stability. This, however, is not essential). The function $v(x) = T^0(x)$, by virtue of system (4.2), satisfies all the conditions of the Liapunov theorem on asymptotic stability [12]. Thus from this point of view for the solution of the problem of optimum control, it is sufficient to find a function $v(x)$, satisfying the conditions of Liapunov's theorem on asymptotic stability, and being such that on the set of admissible control functions $\eta(x)$, $\xi^a(x)$, by virtue of system (1.1), the condition

$$\min(dv/dt) = -1 \tag{4.3}$$

is satisfied.

A Liapunov function $v(x)$ which satisfies these conditions will be called an optimum Liapunov function. From Theorem 3.1 it follows that for Problem II a smooth optimum Liapunov function exists (for Problem I such everywhere smooth optimum Liapunov function $v(x)$ may not exist). It should be emphasized, however, that an effective determination of such a function $v(x)$ is difficult.

Let us formulate now the basic result of this paragraph.

Theorem 4.1. If the conditions (1.10) and (3.3) are satisfied, then for Problem II the optimum control quantities η_0, ξ_0^a, by virtue of system (3.2), are continuous functions of their arguments $x_1, \ldots, x_n, \theta$ which can be differentiated an arbitrary number of times for all $x \neq 0$, $\theta \geqslant 0$*.

Proof. The validity of Theorem 4.1 can be established on the basis of formulas (2.31), (2.32) and Theorem 3.1 on the differentiability of the quantity $T^0(x, \theta)$, since obviously, for the computation of $\eta_0(x_{10}, \ldots, x_{n0})$, $\xi_0^a(x_{10}, \ldots, x_{n0})$, the substitution $t = 0$ must be made in the formulas (2.31) and (2.32). However, we shall also indicate here another method for the proof of Theorem 4.1, which is not based on formula (2.31). Let us present this proof.

According to the arguments mentioned above in this section the optimum control functions $\eta_0(x_1, \ldots, x_n, \theta)$, $\xi_0^a(x_1, \ldots, x_n, \theta)$ can be determined from the condition

$$\left(\frac{dT}{dt}\right)^\circ = \sum_{\mu,\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta}\left(\vartheta\alpha_{\beta\mu}x_\mu + b_\beta\eta_0(x,\vartheta) + \sum_{\alpha=1}^{n-1} e_\beta{}^\alpha \xi_0{}^\alpha(x,\vartheta)\right) =$$

$$= \min \sum_{\mu,\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta}\left(\vartheta a_{\beta\mu}x_\mu + b_\beta\eta(x,\vartheta) + \sum_{\alpha=1}^{n} e_\beta{}^\alpha \xi^\alpha(x,\vartheta)\right) \tag{4.4}$$

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* See the remark at the end of Section 3.

under the assumption

$$\eta^2(x, \vartheta) + \sum_{\alpha=1}^{n-1} [\xi^\alpha(x, \vartheta)]^2 \leqslant 1 \tag{4.5}$$

The solutions of the problems (4.4) and (4.5), obviously, have the form

$$\eta_0(x, \vartheta) = -\sum_{\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta} b_\beta \left[\left(\sum_{\beta=1}^{n} \frac{\partial T}{\partial x_\beta} b_\beta\right)^2 + \sum_{\alpha=1}^{n-1}\left(\sum_{\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta} e_\beta{}^\alpha\right)^2\right]^{-1/2} \tag{4.6}$$

$$\xi_0{}^\alpha(x, \vartheta) = -\sum_{\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta} e_\beta{}^\alpha \left[\left(\sum_{\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta} b_\beta\right)^2 + \sum_{\alpha=1}^{n-1}\left(\sum_{\beta=1}^{n} \frac{\partial T^\circ}{\partial x_\beta} e_\beta{}^\alpha\right)^2\right]^{-1/2} \tag{4.7}$$

The matrix (3.1) is nonsingular and the vector gradient $\{\partial T^0/\partial x_\beta\}$ of the Liapunov function $v(x) = T^0(x, \theta)$ is different from zero. Therefore, as a consequence of the differentiability of the function $T^0(x, \theta)$ (Theorem 3.1), we conclude from the formulas (4.6) and (4.7), that the optimum control functions $\eta_0(x, \theta)$ and $\xi_0{}^\alpha(x, \theta)$ are continuously differentiable, an arbitrary number of times, with respect to all their arguments. This proves the theorem.

Remark. Equation (3.4) and equalities (4.6), (4.7) allow us to solve the problem of optimum control by reduction of this problem to usual variational problems. However, the difficulties which arise here in connection with the corresponding calculations are very great. This fact makes an effective determination of the optimum Liapunov function $v = T^0(x_1, \ldots, x_n)$ and, consequently, of the functions $\eta_0(x)$, $\xi_0{}^\alpha(x)$, difficult.

For an approximate construction of the optimum system in the case of Problem II the following method can be used. Let $v_0(x)$ be a positive-definite Liapunov function for the system (1.9), which has by virtue of this system a negative derivative. If the conditions (1.10) are satisfied, then such a function $v_0(x)$ exists and can be selected in the form of a positive quadratic form.

Let us calculate the derivative dv_0/dt by virtue of system (1.1) and let us select the functions $\eta_1(x)$, $\xi_1^\alpha(x)$ from the conditions

$$\sum_{\beta=1}^{n} \frac{\partial v_0}{\partial x_\beta}\left(b_\beta \eta_1(x) + \sum_{\alpha=1}^{n-1} e_\beta{}^\alpha \xi_1{}^\alpha(x)\right) = \min \tag{4.8}$$

$$\eta_1{}^2(x) + \sum_{\alpha=1}^{n} [\xi_1{}^\alpha(x)]^2 = 1 \tag{4.9}$$

i.e.

$$\eta_1(x) = -\sum_{\beta=1}^{n} \frac{\partial v_0}{\partial x_\beta} b_\beta \left[\left(\sum_{\beta=1}^{n} \frac{\partial v_0}{\partial x_\beta} b_\beta\right)^2 + \sum_{\alpha=1}^{n-1}\left(\sum_{\beta=1}^{n} \frac{\partial v_0}{\partial x_\beta} e_\beta^{\alpha}\right)^2\right]^{-1/2} \qquad (4.10)$$

$$\xi_1^{\alpha}(x) = -\sum_{\beta=1}^{n} \frac{\partial v_0}{\partial x_\beta} b_\beta \left[\left(\sum_{\beta=1}^{n} \frac{\partial v_0}{\partial x_\beta} b_\beta\right)^2 + \sum_{\alpha=1}^{n-1}\left(\sum_{\beta=1}^{n} \frac{\partial v_0}{\partial x_\beta} e_\beta^{\alpha}\right)^2\right]^{-1/2} \qquad (4.11)$$

After the substitution of $\eta = \eta_1$, $\xi^a = \xi_1^a$, as given by (4.10), (4.11), into equations (1.1), we obtain a system which is asymptotically stable on the whole. For this system there exists a Liapunov function $v_1(x)$, satisfying the condition

$$dv_1(x) / dt = -1 \qquad (4.12)$$

The existence of the function $v_1(x)$ can be proved by the methods of inversion of the Liapunov theorems [13] (The fact that here $x(t) \to 0$ as $t \to T^0$, and not as $t \to \infty$, does not play a decisive role in the problem considered concerning the existence of $v_1(x)$). The existence theorems for Liapunov's functions do not give effective methods for the construction of these functions. Assume, however, that we succeeded in constructing a smooth function, the derivative of which by virtue of system (1.1) satisfies the condition

$$dv_1 / dt \approx -1 \qquad (4.13)$$

Compute anew the derivative of the function $v_1(x)$ by virtue of system (1.1), where $\eta = \eta_2$, $\xi^a = \xi_2^a$, and determine these function η_2, ξ_2^a from the condition

$$dv_1 / dt = \min \qquad (4.14)$$

for

$$\eta_2^2 + \sum_{\alpha=1}^{n-1} [\xi_2^{\alpha}]^2 = 1 \qquad (4.15)$$

and so on. If in the kth step we had succeeded in an effective construction of a smooth Liapunov function $v_k(x)$ which by virtue of the system of equations, constructed in the immediately preceding step, reasonably well approximates the condition

$$dv_k / dt = -1 \qquad (4.16)$$

then after a certain number of steps we would obtain a system of equations possessing good optimum properties.

Unfortunately, at present it is impossible to indicate such a general effective method for the construction of a smooth function $v_k(x)$, satisfying completely by virtue (or approximating reasonably well) of a known system of equations the condition (4.16). One of the methods for such an approximation may consist in finding the functions $v_k(x)$ in the form of an expansion according to certain functions (for example, trigonometric polynomials), approximating the condition (4.16) in the mean. However, this method also leads to cumbersome computations.

5. In this section the limit passage from the solutions of Problem II to the solutions of Problem I is investigated for $e_\beta^\alpha \to 0$ ($\alpha = 1, \ldots, n-1$; $\beta = 1, \ldots, n$).

Theorem 5.1. If the conditions (1.10), (1.12) and (3.3) are satisfied, then for all x the optimum control time $T_{II}^0(s_1, \ldots, x_n)$ of Problem II converges to the optimum control time $T_I^0(x_1, \ldots, x_n)$ of Problem I for $e_\beta^\alpha \to 0$, i.e.

$$\lim T_{II}^\circ(x_1, \ldots, x_n) = T_I^\circ(x_1, \ldots, x_n), \qquad \sum_{\alpha,\beta} [e_\beta^\alpha]^2 \to 0 \tag{5.1}$$

holds.

Proof. According to the results mentioned in Section 2, the optimum time $T_I(x_1, \ldots, x_n)$ is to be calculated from equation (2.30), while the optimum time $T_{II}(x_1, \ldots, x_n)$ from equation (2.32).

It follows from these equations that

$$T_{II}(x_1, \ldots, x_n) \leqslant T_I(x_1, \ldots, x_n) \tag{5.2}$$

On the other hand, it is obvious that for $t = T^* = T_I^0 - T_{II}^0$ (where $\Delta T > 0$) we have

$$\min \int_0^{T^*} \Big| \sum_{\beta=1}^n l_\beta h_\beta(\tau) \Big| d\tau < 1 \qquad \Big(\sum_{\beta=1}^n l_\beta x_\beta = -1\Big) \tag{5.3}$$

and

$$\lim \min \int_0^{T^*} \Big[\Big(\sum_{\beta=1}^n l_\beta h_\beta(\tau)\Big)^2 + \Big(\sum_{\beta=1}^n l_\beta g_\beta^1(\tau)\Big)^2 + \cdots + \Big(\sum_{\beta=1}^n l_\beta g_\beta^{n-1}(\tau)\Big)^2\Big]^{1/2} d\tau =$$

$$= \min \int_0^{T^*} \Big| \sum_{\beta=1}^n l_\beta h_\beta(\tau) \Big| d\tau \quad \text{for } e_\beta^\alpha \to 0 \qquad \Big(\sum_{\beta=1}^n l_\beta x_\beta = -1\Big)$$

i.e. for sufficiently small values of e_β^α we have the inequality

$$\min \int_0^{T^*} \Big[\Big(\sum_{\beta=1}^n l_\beta h_\beta(\tau)\Big)^2 + \Big(\sum_{\beta=1}^n l_\beta g_\beta^1(\tau)\Big)^2 + \cdots + \Big(\sum_{\beta=1}^n l_\beta g_\beta^{n-1}(\tau)\Big)^2\Big]^{1/2} d\tau < 1$$

$$\Big(\sum_{\beta=1}^n l_\beta x_\beta = -1\Big)$$

From the last inequality and inequality (5.2) we conclude that for sufficiently small values of e_β^a the inequality

$$T_1^\circ - \Delta T \leqslant T_{11}^\circ \leqslant T_1^\circ$$

is satisfied. This proves the theorem.

Remark. Using the condition (5.1), it is possible to verify that for $e_\beta^a \to 0$ we have not only convergence of the optimum time $T_{II}^0 \to T_I^0$ but also convergence in the measure of the optimum control functions $\eta_{0II} \to \eta_{0I}$.

Theorem 5.1 justifies the following method for the determination of optimum control for Problem I: construct an auxiliary system (1.1) with sufficiently small numbers e_β^a, solve Problem II for this system and put $\eta_{0I} = \eta_{0II}$. As it will be shown in the next section, such a method is justified by the fact that for Problem II it is possible to indicate a regular method of solution.

6. In this section a differential equation is derived which allows us to determine the optimum control time $T^0(x_1, \ldots, x_n)$ and the optimum control functions for Problem II*.

Consider anew side by side the systems (1.1) and (3.2), the last system going over into system (1.1) for $\theta = 1$. As it was shown in Section 3, the optimum control time $T^0(x_1, \ldots, x_n, \theta)$ is a continuously differentiable function of the parameter θ. In the notations of Section 3 the optimum time T^0 is to be determined from the condition

$$\min_{l_2, \ldots, l_n} \gamma(x_1, \ldots, x_n, T, \vartheta, l_2, \ldots, l_n) = \gamma(x_1, \ldots, x_n, T, \vartheta, l_2^\circ, \ldots, l_n^\circ) = 1 \qquad (6.1)$$

where the numbers $l_2^0(x, \theta), \ldots, l_n^0(x, \theta)$ which determine the minimum in the equality (6.1), are also continuously differentiable functions of the parameter θ. Let us make use of this condition for the derivation of a system of differential equations, the integration of which will allow us to determine the quantities $T^0(x, \theta)$ and $l_\beta^0(x, \theta)$. In order to abbreviate writing in what follows we shall drop the arguments x_β which are assumed to be fixed. Substituting in the equality (6.1) the solutions $l_\beta^0(\theta)$, we shall obtain the following equation for the determination of the implicit function $T^0(\theta)$:

$$\gamma(T, \vartheta, l_2^\circ(\vartheta), \ldots, l_n^\circ(\vartheta)) = 1 \qquad (6.2)$$

In conformity with the well-known formula for the derivative of an

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* In this section, in conformity with the remark in Section 3, we may also assume that e_β^a are functions of θ, and that $e_\beta^a \to 0$ for $\theta \to 1$.

implicit function, we can write for $dT^0/d\theta$ the equality

$$\frac{dT^\circ}{d\vartheta} = -\left(\frac{\partial\gamma}{\partial\vartheta} + \sum_{\beta=2}^{n} \frac{\partial\gamma}{\partial l_\beta^\circ}\frac{dl_\beta^\circ}{d\vartheta}\right) \Big/ \frac{\partial\gamma}{\partial T} \tag{6.3}$$

The quantities l_β^0 determine the minimum of the quantity γ. Therefore, for $l = l_\beta^0$ the equalities (3.9) are satisfied, i.e. $\partial\gamma/\partial l_\beta^0 = 0$ ($\beta = 2, \ldots, n$). Consequently, the function $T^0(\theta)$ satisfies the differential equation

$$\frac{dT}{d\vartheta} = -\frac{\partial\gamma/\partial\vartheta}{\partial\gamma/\partial T} \tag{6.4}$$

Substituting $T = T^0(\theta)$ into the equality

$$\gamma(T, \vartheta, l_2^\circ, \ldots, l_n^\circ) = \min\gamma(T, \vartheta, l_2, \ldots, l_n) \tag{6.5}$$

we obtain $n-1$ equations for the determination of the implicit functions l_β^0:

$$\Delta_\beta = \frac{\partial\gamma(T^\circ(\vartheta), \vartheta, l_2^\circ, \ldots, l_n^\circ)}{\partial l_\beta^\circ} = 0 \quad (\beta = 2, \ldots, n) \tag{6.6}$$

Therefore, in conformity with the well-known formulas for the differentiation of implicit functions [10], we conclude that the functions $l_\beta^0(\theta)$ must satisfy the system of differential equations

$$\frac{dl_\beta^\circ(\vartheta)}{d\vartheta} = \frac{D(\Delta_2, \ldots, \Delta_n)/D(l_2^\circ, \ldots, \vartheta_\beta, \ldots, l_n^\circ)}{D(\Delta_2, \ldots, \Delta_n)/D(l_2^\circ, \ldots, l_n^\circ)} \qquad (\beta = 2, \ldots, n) \tag{6.7}$$

In computing the functional determinants $D(l_2^0, \ldots, \theta_\beta, \ldots, l_n^0)$ in the numerator of the equality (6.7), it must be taken into account that for the computation of the derivative of Δ_a with respect to θ, the quantity $T^0(\theta)$ in Δ_a is assumed to be a known function of θ, i.e. taking into account the equality (6.4) in the βth column of these determinants the following expressions must be written

$$\frac{\partial\Delta_\alpha}{\partial\vartheta} + \frac{\partial\Delta_\alpha}{\partial T^\circ}\frac{dT^\circ}{d\vartheta} = \frac{\partial\Delta_\alpha}{\partial\vartheta} - \frac{\partial\Delta_\alpha}{\partial T^\circ}\frac{\partial\gamma/\partial\vartheta}{\partial\gamma/\partial T^\circ} \qquad (\alpha = 2, \ldots, n) \tag{6.8}$$

Consequently, for fixed initial values x_β the functions $T^0(\theta)$ and $l_\beta^0(\theta)$ ($\beta = 2, \ldots, n$) satisfy the equation (6.4) and the system (6.7).

The system of equations (6.4), (6.7) permits to indicate the following method for the solution of Problem II (and also of Problem I by replacing it by an auxiliary approximate Problem II)* (see thefootnote on p. 916): determine the solutions $T^0(0)$ and $l_\beta^0(0)$ ($\beta = 2, \ldots, n$) for $\theta = 0$ and integrate the system of equations (6.4), (6.7) for $0 \leqslant \theta \leqslant 1$. The solutions $T^0(1)$, $l_\beta^0(1)$ ($\beta = 2, \ldots, n$) determine the optimum control time T^0 and the optimum control functions η_0, ξ_0^a [according to the formulas

(2.31)]. The solutions $T^0(0)$ and $l_\beta^0(0)$ can be determined very simply from the conditions (2.31) and (2.32), since for $\theta = 0$ the fundamental matrix of solutions $F(t)$ of the system (3.2) is a unit matrix. Equations (6.4), (6.7) have complicated right-hand sides and cannot be integrated in terms of elementary functions. These equations, however, can be integrated by means of any one of the known numerical methods. The solution of equations (6.4), (6.7) requires cumbersome computations. However, this method of solution allows us to circumvent one of the main difficulties in solving problems of optimum control, namely the necessity of solving boundary-value problems.

In order to obtain the quantities T^0 and η_0, ξ_0^a in the form of functions of the coordinates, we can approximate $T^0(x, \theta)$ and $l_\beta^0(x, \theta)$ in a region we are interested in, for the purpose of the synthesis of the system by a system of orthogonal functions, the coefficients of these expansions, being functions of the parameter θ, and derive from system (6.4), (6.7) differential equations for the determination of these coefficients.

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